Q1. Let $x =$ (last digit of your index number)Mod $3 + 1$. Select matrix number x and call it A.

$$\begin{pmatrix} -11 & -10 & 5 \\ 5 & 4 & -5 \\ -20 & -20 & 4 \end{pmatrix}, \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$$

You will be given the characteristic polynomial, eigenvalues and eigenvectors.

Diagonalize the matix A. Find $A^{10}, A^{-1}, A^{\frac{1}{2}}$and e^{A}.

Solution: Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. Eigenvactors corresponding to $\lambda = -2$ are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Eigenvector corresponding to $\lambda = 4$ is $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ (See Soluitions to Test8)

So we have the diagonalization $AP = P\Lambda$ with $P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Here P is invertible(why?) and $P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -2 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

Now $A = P\Lambda P^{-1}$ so $A^2 = AA = (P\Lambda P^{-1})(P\Lambda P^{-1}) = P\Lambda(P^{-1}P)\Lambda P^{-1} = P\Lambda I\Lambda P^{-1} = P\Lambda^2 P^{-1}$

Here $\Lambda^2 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} (-2)^2 & 0 & 0 \\ 0 & (-2)^2 & 0 \\ 0 & 0 & 4^2 \end{pmatrix}$

In the same way $A^n = P\Lambda^n P^{-1}$ and $\Lambda^n = \begin{pmatrix} (-2)^n & 0 & 0 \\ 0 & (-2)^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}$: raising the diagonal to power n

So $A^{10} = P\Lambda^{10}P^{-1} =$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} (-2)^{10} & 0 & 0 \\ 0 & (-2)^{10} & 0 \\ 0 & 0 & 4^{10} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -2 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 524800 & -523776 & 523776 \\ 523776 & -5227522 & 523776 \\ 1047552 & -1047552 & 1048576 \end{pmatrix}$$

With $\Lambda^{\frac{1}{2}} = \begin{pmatrix} (-2)^{\frac{1}{2}} & 0 & 0 \\ 0 & (-2)^{\frac{1}{2}} & 0 \\ 0 & 0 & 4^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \pm i\sqrt{2} & 0 & 0 \\ 0 & \pm i\sqrt{2} & 0 \\ 0 & 0 & \pm 2 \end{pmatrix}$ we see that $\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}} = \Lambda$

Also $(P\Lambda^{\frac{1}{2}}P^{-1})(P\Lambda^{\frac{1}{2}}P^{-1}) = P\Lambda^{\frac{1}{2}}I\Lambda^{\frac{1}{2}}P^{-1} = P\Lambda P^{-1} = A$.

So we can define $A^{\frac{1}{2}} = P\Lambda^{\frac{1}{2}}P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} \pm i\sqrt{2} & 0 & 0 \\ 0 & \pm i\sqrt{2} & 0 \\ 0 & 0 & \pm 2 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -2 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ which has 8 values

Also $A^{-1} = (P\Lambda P^{-1})^{-1} = P\Lambda^{-1}P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -2 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1/8 & -3/8 & 3/8 \\ 3/8 & -7/8 & 3/8 \\ 3/4 & -3/4 & 1/4 \end{pmatrix}$

$$e^{\Lambda} = \sum_{n=0}^{\infty}\frac{\Lambda^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix} (-2)^n & 0 & 0 \\ 0 & (-2)^n & 0 \\ 0 & 0 & 4^n \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty}\frac{1}{n!}(-2)^n & 0 & 0 \\ 0 & \sum_{n=0}^{\infty}\frac{1}{n!}(-2)^n & 0 \\ 0 & 0 & \sum_{n=0}^{\infty}\frac{1}{n!}4^n \end{pmatrix} = \begin{pmatrix} e^{-2} & 0 & 0 \\ 0 & e^{-2} & 0 \\ 0 & 0 & e^{4} \end{pmatrix}$$

$$e^{A} = \sum_{n=0}^{\infty}\frac{A^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}P\Lambda^n P^{-1} = P\left(\sum_{n=0}^{\infty}\frac{\Lambda^n}{n!}\right)P^{-1} = Pe^{\Lambda}P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} e^{-2} & 0 & 0 \\ 0 & e^{-2} & 0 \\ 0 & 0 & e^{4} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -2 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Note: The above infinite series of matrices are always converging. We take $A^0 = I$.

Note: Given eigenvalues and eigenvalctors means $A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, A\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, A\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

So we can write

$$A\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right) = \left(-2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \; -2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \; 4\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right) = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right)\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

or $A\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ or $AP = P\Lambda$ is the diagonalization of A

Q2. Prove the Caylay-Hamilton Theorem. Use it to find the inverse of A.

Solution:

$\det(A-\lambda I)\,I = (A-\lambda I)\operatorname{adj}(A-\lambda I)$ because of the general result: $\det C\, I = C\operatorname{adj}C$.
Here $\operatorname{adj}(A-\lambda I)$ is a $n\times n$ matrix having polynomials of λ of degree $n-1$ as entries.
We can write such a matrix as a polynomial of λ of degree $n-1$ with coefficients B_i which are $n\times n$ matrices.

Therefore

$p(A)\,I = \det(A-\lambda I)\,I$
$= (A-\lambda I)\operatorname{adj}(A-\lambda I)$
$= (A-\lambda I)\sum_{i=0}^{n-1} B_i\,\lambda^i$
$= \sum_{i=0}^{n-1} AB_i\,\lambda^i - \sum_{i=0}^{n-1} B_i\,\lambda^{i+1} = AB_0 + \sum_{i=1}^{n-1} AB_i\,\lambda^i - \sum_{i=1}^{n-1} B_{i-1}\,\lambda^i - B_{n-1}\lambda^n$
$= AB_0 + \sum_{i=1}^{n-1}(AB_i - B_{i-1})\,\lambda^i - B_{n-1}\lambda^n$

Let $p(\lambda) = \sum_{i=0}^{n} a_i\lambda^i$ so $p(\lambda)I = \sum_{i=0}^{n} a_i I\lambda^i$
Equating the coefficients of λ^i we get
$a_0 I = AB_0$
$a_i I = AB_i - B_{i-1}\,;\, i = 1,\cdots,n-1$
$a_n I = -B_{n-1}$

Now substituting for coefficients a_i we have

$p(A)$
$= \sum_{i=0}^{n} a_i A^i = \sum_{i=0}^{n} A^i a_i I$
$= a_0 I + \sum_{i=1}^{n-1} A^i a_i I + A^n a_n I$
$= AB_0 + \sum_{i=1}^{n-1} A^i(AB_i - B_{i-1}) - A^n B_{n-1}$
$= AB_0 + \sum_{i=1}^{n-1}(A^{i+1}B_i - A^i B_{i-1}) - A^n B_{n-1}$
$= AB_0 + \sum_{i=1}^{n-1} A^{i+1}B_i - \sum_{i=1}^{n-1} A^i B_{i-1} - A^n B_{n-1}$
$= AB_0 + \sum_{i=1}^{n-2} A^{i+1}B_i + A^n B_{n-1} - AB_0 - \sum_{i=1}^{n-1} A^i B_{i-1} - A^n B_{n-1}$
$= \sum_{i=1}^{n-2} A^{i+1}B_i - \sum_{i=1}^{n-2} B_i A^{i+1}$
$= O$

So we have $p(A) = O$ which is the Caylay-Hamilton Theorem.

For the question given $p(\lambda) = \lambda^3 - 12\lambda - 16$ (see Test8 solutions)
Therefore by Caylay-Hamilton Theorem, we have

$p(A) = A^3 - 12A - 16I = O$

Multiplying both sides by A^{-1} we have
$A^{-1}(A^3 - 12A - 16I) = A^{-1}O$
$A^2 - 12I - 16A^{-1} = O$

Therefore
$16A^{-1} = A^2 - 12I$

$$= \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - 12\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -6 & 6 \\ 6 & -2 & 6 \\ 12 & -12 & 16 \end{pmatrix} - 12\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -6 & 6 \\ 6 & -14 & 6 \\ 12 & -12 & 4 \end{pmatrix}$$

So $A^{-1} = \frac{1}{16}\begin{pmatrix} -2 & -6 & 6 \\ 6 & -14 & 6 \\ 12 & -12 & 4 \end{pmatrix} = \begin{pmatrix} -1/8 & -3/8 & 3/8 \\ 3/8 & -7/8 & 3/8 \\ 3/4 & -3/4 & 1/4 \end{pmatrix}$