1. Prove the Cauchy-Schwarz inequality: $|\langle u, v\rangle| \leq \|u\|\|v\|$ for elements u, v in an inner product space.

Solution

Let $\mathrm{F} = \mathbb{C}$.

If $v = \underline{0}$ ie if $\|v\| = 0$ we have $|\langle u, v\rangle| = 0 \leq 0 = \|u\|\|v\|$

Assume $v \neq \underline{0}$ and let $w = u - \frac{\langle u,v\rangle}{\|v\|^2}v$, then

$$
\begin{aligned}
&\|w\|^2 \\
&= \langle w, w\rangle \\
&= \langle u - \tfrac{\langle u,v\rangle}{\|v\|^2}v, u - \tfrac{\langle u,v\rangle}{\|v\|^2}v\rangle \\
&= \langle u, u\rangle - \langle u, \tfrac{\langle u,v\rangle}{\|v\|^2}v\rangle - \langle\tfrac{\langle u,v\rangle}{\|v\|^2}v, u\rangle + \langle\tfrac{\langle u,v\rangle}{\|v\|^2}v, \tfrac{\langle u,v\rangle}{\|v\|^2}v\rangle \\
&= \|u\|^2 - \tfrac{\overline{\langle u,v\rangle}}{\|v\|^2}\langle u, v\rangle - \tfrac{\langle u,v\rangle}{\|v\|^2}\langle v, u\rangle + \tfrac{\langle u,v\rangle}{\|v\|^2}\tfrac{\overline{\langle u,v\rangle}}{\|v\|^2}\langle v, v\rangle \text{ since } \langle ax, by\rangle = a\overline{b}\langle x, y\rangle \\
&= \|u\|^2 - \tfrac{\overline{\langle u,v\rangle}}{\|v\|^2}\langle u, v\rangle - \tfrac{\langle u,v\rangle}{\|v\|^2}\overline{\langle u, v\rangle} + \tfrac{\langle u,v\rangle}{\|v\|^2}\tfrac{\overline{\langle u,v\rangle}}{\|v\|^2}\|v\|^2 \text{ since } \langle x, y\rangle = \overline{\langle y, x\rangle} \\
&= \|u\|^2 - \tfrac{|\langle u,v\rangle|^2}{\|v\|^2} - \tfrac{|\langle u,v\rangle|^2}{\|v\|^2} + \tfrac{|\langle u,v\rangle|^2}{\|v\|^2} \\
&= \|u\|^2 - \tfrac{|\langle u,v\rangle|^2}{\|v\|^2} \geq 0
\end{aligned}
$$

Therefore $|\langle u, v\rangle|^2 \leq \|u\|^2\|v\|^2$ and $|\langle u, v\rangle| \leq \|u\|\|v\|$ (why?)

2. Let $f, g \in \mathcal{C}[a, b]$ and $F = \mathbb{R}$. Show that $\langle f, g\rangle = \int_a^b f(x)g(x)dx$ is an inner product. What does Cauchy-Schwarz inequality look like?

Solution

Let $\mathcal{C}[a, b]$ be the set of Continous Real Valued Fucntions on $[a, b]$
and $f, g, h \in \mathcal{C}[a, b]$ *and* $a \in F = \mathbb{R}$.

1. Now $f, g, fg \in \mathcal{R}[a, b]$ ie Reimann Integrable on $[a, b]$(Theorem, why?)
   hence , $\langle\cdot,\cdot\rangle: \mathcal{C}[a, b] \times \mathcal{C}[a, b] \to \mathbb{R}$ given above is a Function.

2. $\langle f, g\rangle = \int_a^b f(x)g(x)dx = \int_a^b g(x)f(x)dx = \langle g, f\rangle$

3. $\langle f + g, h\rangle = \int_a^b (f + g)(x)h(x)dx = \int_a^b \big(f(x) + g(x)\big)h(x)dx = \int_a^b \big(f(x)h(x) + g(x)h(x)\big)dx$

   $= \int_a^b f(x)h(x)dx + \int_a^b g(x)h(x)dx = \langle f, h\rangle + \langle g, h\rangle$

4. $\langle cf, g\rangle = \int_a^b (cf)(x)g(x)dx = \int_a^b cf(x)g(x)dx = c\int_a^b f(x)g(x)dx = c\langle f, g\rangle$
5. $\langle f, f\rangle = \int_a^b f(x)f(x)dx = \int_a^b \big(f(x)\big)^2 dx \geq 0$ and
   $\langle f, f\rangle = 0 \Leftrightarrow \int_a^b \big(f(x)\big)^2 dx = 0 \Leftrightarrow \forall x \in [a, b], f(x) = 0 = O(x) \Leftrightarrow f = O = \underline{0}$ the Zero Function
   (upto a set of *measure* 0?, why?)

Now $|\langle f, g\rangle| \leq \|f\|\|g\|$ reads as

$$\left|\int_a^b f(x)g(x)dx\right| \leq \sqrt{\int_a^b \big(f(x)\big)^2 dx}\sqrt{\int_a^b (g(x))^2 dx} \text{ or}$$
$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b \big(f(x)\big)^2 dx \int_a^b \big(g(x)\big)^2 dx$$