Solve the following Wave Equation.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, 0 < x < \pi, t > 0$$

$$u(0,t) = 0,\ \ u(\pi,t) = 0,\ \ u(x,0) = \sin x,\ \ \frac{\partial u}{\partial t}(x,0) = x^2$$

Solution by Fourier Series

Let $u(x,t) = X(x)T(t)$

So $u_{xx}(x,t) = X''(x)T(t)$ and $u_{tt}(x,t) = X(x)T''(t)$

We have $u_{tt}(x,t) = u_{xx}(x,t)$ or $X''(x)T(t) = X(x)T''(t)$ or $\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \mu$ where μ is a constant

Let $\mu = -\lambda^2 < 0$(see next page for other cases)

Then $X''(x) = -\lambda^2 X(x)$ and $T''(t) = -\lambda^2 T(t)$

The solutions are $X(x) = a\sin\lambda x + b\cos\lambda x$ and $T(t) = c\sin\lambda t + d\cos\lambda t$

Therefore $u(x,t) = (a\sin\lambda x + b\cos\lambda x)T(t)$

We have $u(0,t) = (b)T(t) \Rightarrow b = 0 \Rightarrow u(x,t) = a\sin\lambda x T(t)$

And $u(\pi,t) = a\sin\lambda\pi T(t) = 0 \Rightarrow \sin\lambda\pi = 0 = \sin n\pi \Rightarrow \lambda\pi = n\pi \Rightarrow \lambda = n \in \mathbb{Z}$

So we have $u(x,t) = a\sin nx T(t) = \sin nx(ac\sin nt + ad\cos nt) = \sin nx(A_n\sin nt + B_n\cos nt)$

Since the PDE is linear following is the general solution

$u(x,t) = \sum_{n=-\infty}^{\infty}\sin nx(A_n\sin nt + B_n\cos nt) = 0 + \sum_{n=1}^{\infty}\sin nx(a_n\sin nt + b_n\cos nt)$

Now $u(x,0) = \sum_{n=1}^{\infty}\sin nx(b_n) = b_1\sin x + \sum_{n=2}^{\infty} b_n\sin nx = \sin x \Rightarrow b_1 = 1$ and $b_n = 0, n \geq 2$

So $u(x,t) = \sum_{n=1}^{\infty} b_n\sin nx\cos nt + \sum_{n=1}^{\infty} a_n\sin nx\sin nt = \sin x\cos t + \sum_{n=1}^{\infty} a_n\sin nx\sin nt$

Now $u_t(x,t) = -\sin x\sin t + \sum_{n=1}^{\infty} na_n\sin nx\cos nt$

And we want $u_t(x,0) = 0 + \sum_{n=1}^{\infty} na_n\sin nx = x^2$

To write x^2 as an infinite sum of using only sin, we will find the Fourier Series of $f(x)$ which is x^2 on $[0,\pi]$ and $-x^2$ on $[-\pi,0]$.

Then we have

$2\pi c_n = \int_{-\pi}^{\pi} f(x)e^{inx}dx = \int_{-\pi}^{0} -x^2e^{inx}dx + \int_0^{\pi} x^2e^{inx}dx = -\int_0^{\pi} x^2e^{-inx}dx + \int_0^{\pi} x^2e^{inx}dx = 2i\mathrm{Im}\int_0^{\pi} x^2e^{inx}dx$

And $2\pi c_0 = 2i\mathrm{Im}\int_0^{\pi} x^2dx = 0 \Rightarrow c_0 = 0$

When $n \neq 0$

$$\int_0^{\pi} x^2e^{inx}dx = \left[x^2\frac{e^{inx}}{in}\right]_0^{\pi} - \int_0^{\pi}\frac{e^{inx}}{in}2xdx = \frac{\pi^2\cos n\pi}{in} - \frac{2}{in}\left[x\frac{e^{inx}}{in} - \frac{e^{inx}}{(in)^2}\right]_0^{\pi}$$

$$= -i\frac{\pi^2\cos n\pi}{n} - \frac{2}{(in)^2}\left[\pi\cos n\pi - \frac{\cos n\pi - 1}{in}\right] = -i\frac{\pi^2(-1)^n}{n} + \frac{2}{n^2}\left[\pi(-1)^n + i\frac{(-1)^n - 1}{n}\right]$$

So $c_n = \frac{1}{\pi}i\mathrm{Im}\int_0^{\pi} x^2e^{inx}xdx = \frac{i}{\pi}\left(-\frac{\pi^2(-1)^n}{n} + \frac{2((-1)^n - 1)}{n^3}\right)$

We also note that $c_{-n} = -c_n$

Then $u_t(x,0) = \sum_{n=1}^{\infty} na_n \sin nx = x^2 = \sum_{n=-\infty}^{\infty} c_n e^{-inx} = c_0 + \sum_{n=1}^{\infty} c_n(e^{-inx} - e^{inx}) = 0 + \sum_{n=1}^{\infty} -c_n 2i\sin nx$

So we need $na_n = -2ic_n$ or $a_n = \frac{-2i}{n}\frac{i}{\pi}\left(-\frac{\pi^2(-1)^n}{n} + \frac{2((-1)^n-1)}{n^3}\right) = -\frac{2\pi(-1)^n}{n^2} - \frac{4(1-(-1)^n)}{\pi n^4}$

And finally $u(x,t) = \sin x \cos t + \sum_{n=1}^{\infty}\left(-\frac{2\pi(-1)^n}{n^2} - \frac{4(1-(-1)^n)}{\pi n^4}\right)\sin nx \sin nt$

This won't be tested, ie. you don't have to show all the cases

Case 1: $\mu = 0$

$X''(x) = 0$ and $T''(t) = 0$,the solutions are $X(x) = ax + b$ and $T(t) = ct + d$

We have $u(x,t) = (ax+b)T(t)$

$u(0,t) = (b)T(t) = 0 \Rightarrow b = 0 \Rightarrow u(x,t) = axT(t)$

$u(\pi,t) = a\pi T(t) = 0 \Rightarrow a = 0 \Rightarrow u(x,t) = 0$,a contradiction

Case 2: $\mu = \lambda^2 > 0$

$X''(x) = \lambda^2 X(x)$ and $T''(t) = \lambda^2 T(t)$,the solutions are $X(x) = ae^{\lambda x} + be^{-\lambda x}$ and $T(t) = ce^{\lambda t} + de^{-\lambda t}$

We have $u(x,t) = \left(ae^{\lambda x} + be^{-\lambda x}\right)T(t)$

$u(0,t) = (a+b)T(t) = 0 \Rightarrow a + b = 0 \Rightarrow b = -a \Rightarrow u(x,t) = a\left(e^{\lambda x} - e^{-\lambda x}\right)T(t)$

$u(\pi,t) = a\left(e^{\lambda\pi} - e^{-\lambda\pi}\right)T(t) = 0 \Rightarrow a = 0 \Rightarrow u(x,t) = 0$,a contradiction

Solution by Laplace Transform (this won't be tested)

Let $(x,\hat{s}) = \int_0^\infty u(x,t)e^{-st}dt$, $u(\hat{k},t) = \int_0^\infty u(x,t)e^{-kx}dx$

$u(0,\hat{s}) = \int_0^\infty u(0,t)e^{-st}dt = 0, u(\pi,\hat{s}) = \int_0^\infty u(\pi,t)e^{-st}dt = 0$

$u(\hat{k},0) = \int_0^\infty u(x,0)e^{-kx}dx = \int_0^\infty \sin x e^{-kx}dx = \frac{1}{1+k^2}$

$u_t(\hat{k},0) = \int_0^\infty u_t(x,0)e^{-kx}dx = \int_0^\infty x^2 e^{-kx}dx = \frac{2}{k^3}$

$u_x(0,\hat{s}) = \int_0^\infty u_x(0,t)e^{-st}dt = g(s)$, say

$u(\hat{k},\hat{s}) = \int_0^\infty u(x,\hat{s})e^{-kx}dx = \int_0^\infty u(\hat{k},t)e^{-st}dt$

Taking Laplace Transform of $u_{tt}(x,t) = u_{xx}(x,t)$ w.r.t t we find that

$\int_0^\infty u_{xx}(x,t)e^{-st}dt = \int_0^\infty u_{tt}(x,t)e^{-st}dt$

Or $\frac{d^2}{dx^2}u(x,\hat{s}) = s^2u(x,\hat{s}) - su(x,0) - u_t(x,0)$, now taking Laplace Transform w.r.t x

We have $k^2u(\hat{k},\hat{s}) - ku(0,\hat{s}) - u_x(0,\hat{s}) = s^2u(\hat{k},\hat{s}) - su(\hat{k},0) - u_t(\hat{k},0)$

And substituting the boundary conditions we get

$k^2u(\hat{k},\hat{s}) - k0 - g(s) = s^2u(\hat{k},\hat{s}) - s\frac{1}{1+k^2} - \frac{2}{k^3}$

$(k^2 - s^2)u(\hat{k},\hat{s}) = -s\frac{1}{1+k^2} - \frac{2}{k^3} + g(s)$

$u(\hat{k},\hat{s}) = -s\frac{1}{(1+k^2)(k^2-s^2)} - \frac{2}{k^3(k^2-s^2)} + \frac{g(s)}{(k^2-s^2)}$

$= -s\left(\frac{-1}{1+k^2} + \frac{1}{k^2-s^2}\right)\frac{1}{1+s^2} - \frac{2}{k}\left(\frac{-1}{k^2} + \frac{1}{k^2-s^2}\right)\frac{1}{s^2} + \frac{g(s)}{(k^2-s^2)}$

$= \frac{s}{1+s^2}\frac{1}{1+k^2} - \frac{s}{1+s^2}\frac{1}{k^2-s^2} + \frac{1}{s^2}\frac{2}{k^3} - \frac{2}{s^2}\frac{1}{k+s}\left(\frac{-1}{k} + \frac{1}{k-s}\right)\frac{1}{s} + \frac{g(s)}{(k^2-s^2)}$

$= \frac{s}{1+s^2}\frac{1}{1+k^2} + \frac{1}{s^2}\frac{2}{k^3} + \left(g(s) - \frac{s}{1+s^2} - \frac{2}{s^3}\right)\left(\frac{-1}{k+s} + \frac{1}{k-s}\right)\frac{1}{2s} + \frac{2}{s^3}\left(\frac{1}{k} + \frac{-1}{k+s}\right)\frac{1}{s}$

$= \frac{s}{1+s^2}\frac{1}{1+k^2} + \frac{1}{s^2}\frac{2}{k^3} + \frac{2}{s^4}\frac{1}{k} + \left(\frac{g(s)}{2s} - \frac{1}{2(1+s^2)} - \frac{1}{s^4}\right)\frac{1}{k-s} + \left(-\frac{g(s)}{2s} + \frac{1}{s(1+s^2)} - \frac{1}{s^4}\right)\frac{1}{k+s}$

$= \frac{s}{1+s^2}\frac{1}{1+k^2} + \frac{1}{s^2}\frac{2}{k^3} + \frac{2}{s^4}\frac{1}{k} + A(s)\frac{1}{k-s} + B(s)\frac{1}{k+s}$ where $A(s) + B(s) = -\frac{2}{s^4} \cdots (1)$

So $u(x,\hat{s}) = \frac{s}{1+s^2}\sin x + \frac{1}{s^2}x^2 + \frac{2}{s^4} + A(s)e^{xs} + B(s)e^{-xs}$

Then $u(\pi,\hat{s}) = 0 + \frac{1}{s^2}\pi^2 + \frac{2}{s^4} + A(s)e^{\pi s} + B(s)e^{-\pi s} = 0$

Or $A(s)e^{\pi s} + B(s)e^{-\pi s} = -\frac{\pi^2}{s^2} - \frac{2}{s^4} \cdots (2)$

$(2) - (1)e^{-\pi s}$: $A(s)(e^{\pi s} - e^{-\pi s}) = A(s)2\sinh\pi s = -\frac{\pi^2}{s^2} - \frac{2}{s^4} + \frac{2}{s^4}e^{-\pi s}$

Therefore $A(s) = \frac{1}{2\sinh\pi s}\left(-\frac{\pi^2}{s^2} - \frac{2}{s^4} + \frac{2}{s^4}e^{-\pi s}\right)$

We have $u(x,\hat{s}) = \frac{s}{1+s^2}\sin x + \frac{1}{s^2}x^2 + \frac{2}{s^4} + A(s)e^{xs} + \left(-A(s) - \frac{2}{s^4}\right)e^{-xs}$

$u(x,\hat{s}) = \frac{s}{1+s^2}\sin x + \frac{1}{s^2}x^2 + \frac{2}{s^4} + A(s)2\sinh xs - \frac{2}{s^4}e^{-xs}$

Or $u(x,t) = \cos t \sin x + tx^2 + \frac{t^3}{3} + \mathfrak{L}^{-1}\{f(s)\}$

Where $f(s) = A(s)2\sinh xs - \frac{2}{s^4}e^{-xs} = \frac{\sinh xs}{\sinh \pi s}\left(-\frac{\pi^2}{s^2} - \frac{2}{s^4} + \frac{2}{s^4}e^{-\pi s}\right) - \frac{2}{s^4}e^{-xs}$

We will use the Complex Inversion Formula to find the inverse Laplace Transform

Ie. $\mathfrak{L}^{-1}\{f(s)\} = \sum_p \text{Res}[f(s)e^{st}, p]$, p are poles of $f(s)e^{st}$

For the poles, we solve, $\sinh \pi s = \frac{e^{\pi s} - e^{-\pi s}}{2} = 0 \Leftrightarrow e^{2\pi s} = 1 = e^{i2\pi n}, n \in \mathbb{Z}$

So we have poles at $s = in \in i\mathbb{Z}$. The pole at $s = 0$ is of order 4 and the other poles are simple(order 1)

When $n \neq 0$,

$\text{Res}[f(s)e^{st}, in] = \lim_{s\to in}(s - in)\left[\frac{\sinh xs}{\sinh \pi s}\left(-\frac{\pi^2}{s^2} - \frac{2}{s^4} + \frac{2}{s^4}e^{-\pi s}\right) - \frac{2}{s^4}e^{-xs}\right]e^{st}$

$= \sinh(xin)\left(-\frac{\pi^2}{(in)^2} - \frac{2}{(in)^4} + \frac{2}{(in)^4}e^{-\pi(in)}\right)e^{int}\lim_{s\to in}\frac{s-in}{\sinh \pi s} - \frac{2}{(in)^4}e^{in(t-x)}\lim_{s\to in}(s - in)$

$= i\sin(nx)\left(\frac{\pi^2}{n^2} - \frac{2}{n^4} + \frac{2}{n^4}(-1)^n\right)e^{int}\frac{1}{\pi(-1)^n} + 0$, since $\lim_{s\to in}\frac{s-in}{\sinh \pi s} = \lim_{s\to in}\frac{1}{\pi\cosh \pi s} = \frac{1}{\pi\cosh(i\pi n)} = \frac{1}{\pi\cos(\pi n)} = \frac{1}{\pi(-1)^n}$

$= \left(\frac{\pi(-1)^n}{n^2} + \frac{2(1-(-1)^n)}{\pi n^4}\right)i\sin nxe^{int}$

And

$\text{Res}[f(s)e^{st}, 0] = \lim_{s\to 0}\frac{1}{3!}\frac{d^3}{ds^3}\left\{s^4\left[\frac{\sinh xs}{\sinh \pi s}\left(-\frac{\pi^2}{s^2} - \frac{2}{s^4} + \frac{2}{s^4}e^{-\pi s}\right) - \frac{2}{s^4}e^{-xs}\right]e^{st}\right\}$

$= \lim_{s\to 0}\frac{1}{3!}\frac{d^3}{ds^3}\left\{\frac{\sinh xs}{\sinh \pi s}(-\pi^2 s^2 - 2 + 2e^{-\pi s})e^{st} - 2e^{s(t-x)}\right\}$

We will use MATHEMATICA for this, see next page

The answer is, $\text{Res}[f(s)e^{st}, 0] = -\frac{t^3}{3} - tx^2$

Therefore

$\mathfrak{L}^{-1}\{f(s)\} = \sum_p \text{Res}[f(s)e^{st}, p] = \text{Res}[f(s)e^{st}, 0] + \sum_{n=-\infty, n\neq 0}^{\infty}\text{Res}[f(s)e^{st}, in]$

$= -\frac{t^3}{3} - tx^2 + \sum_{n=-\infty, n\neq 0}^{\infty}\left(\frac{\pi(-1)^n}{n^2} + \frac{2(1-(-1)^n)}{\pi n^4}\right)i\sin nxe^{int}$

$= -\frac{t^3}{3} - tx^2 + \sum_{n=1}^{\infty}\left(\frac{\pi(-1)^n}{n^2} + \frac{2(1-(-1)^n)}{\pi n^4}\right)i\sin nx(e^{int} - e^{-int})$

$= -\frac{t^3}{3} - tx^2 + \sum_{n=1}^{\infty}\left(\frac{\pi(-1)^n}{n^2} + \frac{2(1-(-1)^n)}{\pi n^4}\right)i\sin nx2i\sin nt$

$= -\frac{t^3}{3} - tx^2 + \sum_{n=1}^{\infty}\left(-\frac{2\pi(-1)^n}{n^2} - \frac{4(1-(-1)^n)}{\pi n^4}\right)\sin nx\sin nt$

Finally $u(x,t) = \cos t\sin x + tx^2 + \frac{t^3}{3} - \frac{t^3}{3} - tx^2 + \sum_{n=1}^{\infty}\left(-\frac{2\pi(-1)^n}{n^2} - \frac{4(1-(-1)^n)}{\pi n^4}\right)\sin nx\sin nt$

Or $u(x,t) = \sin x\cos t + \sum_{n=1}^{\infty}\left(-\frac{2\pi(-1)^n}{n^2} - \frac{4(1-(-1)^n)}{\pi n^4}\right)\sin nx\sin nt$

There are some places to fix, see if you can find them. It was a good mistake trying to solve this by Laplace!

In[1]:= $$f[s\_,x\_] := \frac{\text{Sinh}[x\,s]}{\text{Sinh}[\pi\,s]}\left(-\frac{\pi^2}{s^2}-\frac{2}{s^4}+\frac{2}{s^4}e^{-\pi s}\right)-\frac{2}{s^4}e^{-x s}$$

In[2]:= $$\text{Expand}\left[D\left[s\text{^}4\,f[s,x]\,e^{s t},\{s,3\}\right]/6\right]$$

Out[2]=
$$-\frac{1}{3}e^{st-sx}t^3+e^{st-sx}t^2x-e^{st-sx}t\,x^2+\frac{1}{3}e^{st-sx}x^3-e^{st}\pi^2x\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]+$$
$$e^{-\pi s+st}\pi^2x\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]-2e^{-\pi s+st}\pi t\,x\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]-2e^{st}\pi^2s\,t\,x\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]-$$
$$e^{st}t^2x\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]+e^{-\pi s+st}t^2x\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]-\frac{1}{2}e^{st}\pi^2s^2t^2x\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]-$$
$$\frac{1}{3}e^{st}x^3\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]+\frac{1}{3}e^{-\pi s+st}x^3\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]-\frac{1}{6}e^{st}\pi^2s^2x^3\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]+$$
$$2e^{-\pi s+st}\pi^2x\,\text{Cosh}[s\,x]\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]+2e^{st}\pi^3s\,x\,\text{Cosh}[s\,x]\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]+$$
$$2e^{st}\pi t\,x\,\text{Cosh}[s\,x]\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]-2e^{-\pi s+st}\pi t\,x\,\text{Cosh}[s\,x]\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]+$$
$$e^{st}\pi^3s^2t\,x\,\text{Cosh}[s\,x]\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]-e^{st}\pi^2x\,\text{Cosh}[s\,x]\,\text{Coth}[\pi s]^2\,\text{Csch}[\pi s]+$$
$$e^{-\pi s+st}\pi^2x\,\text{Cosh}[s\,x]\,\text{Coth}[\pi s]^2\,\text{Csch}[\pi s]-\frac{1}{2}e^{st}\pi^4s^2x\,\text{Cosh}[s\,x]\,\text{Coth}[\pi s]^2\,\text{Csch}[\pi s]-$$
$$e^{st}\pi^2x\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]^3+e^{-\pi s+st}\pi^2x\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]^3-\frac{1}{2}e^{st}\pi^4s^2x\,\text{Cosh}[s\,x]\,\text{Csch}[\pi s]^3-$$
$$\frac{1}{3}e^{-\pi s+st}\pi^3\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-e^{st}\pi^2t\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+e^{-\pi s+st}\pi^2t\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-$$
$$e^{-\pi s+st}\pi t^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-e^{st}\pi^2s\,t^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-\frac{1}{3}e^{st}t^3\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+$$
$$\frac{1}{3}e^{-\pi s+st}t^3\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-\frac{1}{6}e^{st}\pi^2s^2t^3\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-e^{-\pi s+st}\pi x^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-$$
$$e^{st}\pi^2s\,x^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-e^{st}t\,x^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+e^{-\pi s+st}t\,x^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-$$
$$\frac{1}{2}e^{st}\pi^2s^2t\,x^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+e^{st}\pi^3\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-$$
$$e^{-\pi s+st}\pi^3\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+2e^{-\pi s+st}\pi^2t\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+$$
$$2e^{st}\pi^3s\,t\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+e^{st}\pi t^2\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-$$
$$e^{-\pi s+st}\pi t^2\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+\frac{1}{2}e^{st}\pi^3s^2t^2\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+$$
$$e^{st}\pi x^2\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-e^{-\pi s+st}\pi x^2\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+$$
$$\frac{1}{2}e^{st}\pi^3s^2x^2\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-e^{-\pi s+st}\pi^3\,\text{Coth}[\pi s]^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-$$
$$e^{st}\pi^4s\,\text{Coth}[\pi s]^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-e^{st}\pi^2t\,\text{Coth}[\pi s]^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+$$
$$e^{-\pi s+st}\pi^2t\,\text{Coth}[\pi s]^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-\frac{1}{2}e^{st}\pi^4s^2t\,\text{Coth}[\pi s]^2\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+$$
$$\frac{1}{3}e^{st}\pi^3\,\text{Coth}[\pi s]^3\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-\frac{1}{3}e^{-\pi s+st}\pi^3\,\text{Coth}[\pi s]^3\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]+$$
$$\frac{1}{6}e^{st}\pi^5s^2\,\text{Coth}[\pi s]^3\,\text{Csch}[\pi s]\,\text{Sinh}[s\,x]-e^{-\pi s+st}\pi^3\,\text{Csch}[\pi s]^3\,\text{Sinh}[s\,x]-$$
$$e^{st}\pi^4s\,\text{Csch}[\pi s]^3\,\text{Sinh}[s\,x]-e^{st}\pi^2t\,\text{Csch}[\pi s]^3\,\text{Sinh}[s\,x]+e^{-\pi s+st}\pi^2t\,\text{Csch}[\pi s]^3\,\text{Sinh}[s\,x]-$$
$$\frac{1}{2}e^{st}\pi^4s^2t\,\text{Csch}[\pi s]^3\,\text{Sinh}[s\,x]+\frac{5}{3}e^{st}\pi^3\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]^3\,\text{Sinh}[s\,x]-$$
$$\frac{5}{3}e^{-\pi s+st}\pi^3\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]^3\,\text{Sinh}[s\,x]+\frac{5}{6}e^{st}\pi^5s^2\,\text{Coth}[\pi s]\,\text{Csch}[\pi s]^3\,\text{Sinh}[s\,x]$$

In[3]:= **Expand[Limit[%, s → 0]]**

Out[3]= $$-\frac{t^3}{3}-t\,x^2$$

Or directly using Residue

In[4]:= **Expand[Residue[f[s, x] E^(s t), {s, 0}]]**

Out[4]= $-\frac{t^3}{3} - t\,x^2$

See that directly using InverseLaplaceTransform don ' t give any results

In[5]:= **InverseLaplaceTransform[f[s, x], s, t]**

Out[5]= $\text{InverseLaplaceTransform}\left[\frac{2\left(-\frac{1}{2}e^{-s x}+\frac{e^{s x}}{2}\right)\left(-\frac{2}{s^4}+\frac{2 e^{-\pi s}}{s^4}-\frac{\pi^2}{s^2}\right)}{-e^{-\pi s}+e^{\pi s}}-\frac{2 e^{-s x}}{s^4}, s, t\right]$

Note that Dsolve can ' t sove this PDE with boundary conditions

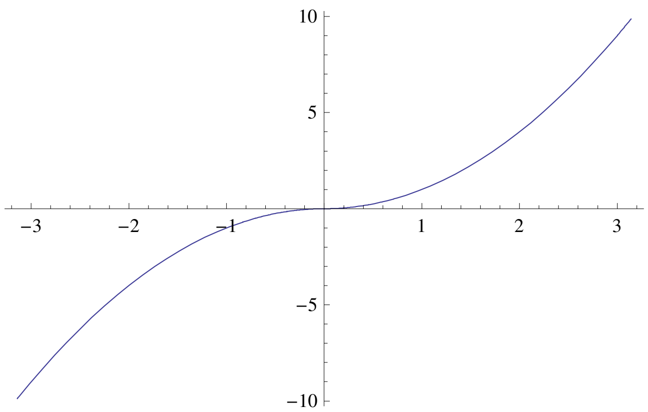
In[6]:= **DSolve[{D[u[x, t], {t, 2}] == D[u[x, t], {x, 2}], u[0, t] == 0, u[Pi, t] == 0, u[x, 0] == Sin[x], x^2 == D[u[x, t], t] /. t → 0}, u[x, t], x, t]**

Out[6]= $\text{DSolve}\left[\left\{u^{(0,2)}[x,t]==u^{(2,0)}[x,t], u[0,t]==0, u[\pi,t]==0, u[x,0]==\text{Sin}[x], x^2==u^{(0,1)}[x,0]\right\}, u[x,t], x, t\right]$

But you can find Fourier Series

In[7]:= **h[x_] := Piecewise[{{-x^2, x < 0}, {x^2, x ≥ 0}}]**

In[8]:= **Plot[h[x], {x, -Pi, Pi}]**

Out[8]=


In[9]:= **FourierSeries[h[x], x, 5]**

Out[9]= $e^{5 i x}\left(\frac{4 i}{125\pi}-\frac{i\pi}{5}\right)-\frac{1}{2} i e^{-2 i x}\pi+\frac{1}{2} i e^{2 i x}\pi-\frac{1}{4} i e^{-4 i x}\pi+\frac{1}{4} i e^{4 i x}\pi+\frac{i e^{-i x}\left(-4+\pi^2\right)}{\pi}-\frac{i e^{i x}\left(-4+\pi^2\right)}{\pi}+\frac{i e^{-3 i x}\left(-4+9\pi^2\right)}{27\pi}-\frac{i e^{3 i x}\left(-4+9\pi^2\right)}{27\pi}+\frac{i e^{-5 i x}\left(-4+25\pi^2\right)}{125\pi}$

In[10]:= **Table[FourierCoefficient[h[x], x, n], {n, 0, 5}]**

Out[10]= $\left\{0, -\frac{i\left(-4+\pi^2\right)}{\pi}, \frac{i\pi}{2}, -\frac{i\left(-4+9\pi^2\right)}{27\pi}, \frac{i\pi}{4}, \frac{4 i}{125\pi}-\frac{i\pi}{5}\right\}$

Lets see if we have solved the problem

In[11]:= $\mathbf{b[n\_] := -\frac{2\,Pi\,(-1)\hat{}n}{n\hat{}2} - \frac{4\,(1-(-1)\hat{}n)}{Pi\,n\hat{}4}}$

In[12]:= $\mathbf{u[x\_, t\_] := Sin[x]\,Cos[t] + \sum_{n=1}^{\infty} b[n]\,Sin[n\,x]\,Sin[n\,t]}$

In[13]:= **D[u[x, t], {t, 2}] - D[u[x, t], {x, 2}]**

Out[13]= 0

In[14]:= **u[0, t]**

Out[14]= 0

In[15]:= `u[Pi, t]`

Out[15]= 0

In[16]:= `u[x, 0]`

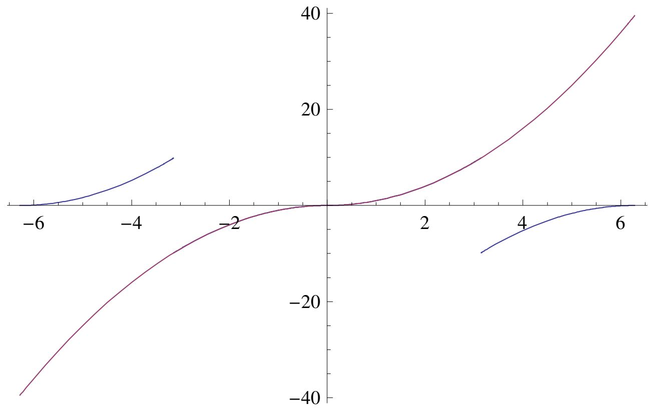
Out[16]= `Sin[x]`

In[17]:= `D[u[x, t], t] /. t → 0`

In[18]:= $k[x\_] := \frac{1}{2\pi}\left(2\,i\,\pi^2\,\mathrm{Log}\left[1 + e^{-i\,x}\right] - 2\,i\,\pi^2\,\mathrm{Log}\left[1 + e^{i\,x}\right] + 4\,i\,\mathrm{PolyLog}\left[3, -e^{-i\,x}\right] - 4\,i\,\mathrm{PolyLog}\left[3, e^{-i\,x}\right] - 4\,i\,\mathrm{PolyLog}\left[3, -e^{i\,x}\right] + 4\,i\,\mathrm{PolyLog}\left[3, e^{i\,x}\right]\right)$

In[19]:= `Plot[{k[x], h[x]}, {x, -2 Pi, 2 Pi}]`

Out[19]=



Lets plot the solution in 3 D

In[20]:= $\mathrm{Sin}[x]\,\mathrm{Cos}[t] + \sum_{n=1}^{\infty} b[n]\,\mathrm{Sin}[n\,x]\,\mathrm{Sin}[n\,t]$

In[21]:= $v[x\_, t\_] := \frac{1}{2\pi}\left(-\pi^2\,\mathrm{PolyLog}\left[2, -e^{-i\,(t-x)}\right] - \pi^2\,\mathrm{PolyLog}\left[2, -e^{i\,(t-x)}\right] + \pi^2\,\mathrm{PolyLog}\left[2, -e^{-i\,(t+x)}\right] + \pi^2\,\mathrm{PolyLog}\left[2, -e^{i\,(t+x)}\right] + 2\,\mathrm{PolyLog}\left[4, -e^{-i\,(t-x)}\right] - 2\,\mathrm{PolyLog}\left[4, e^{-i\,(t-x)}\right] + 2\,\mathrm{PolyLog}\left[4, -e^{i\,(t-x)}\right] - 2\,\mathrm{PolyLog}\left[4, e^{i\,(t-x)}\right] - 2\,\mathrm{PolyLog}\left[4, -e^{-i\,(t+x)}\right] + 2\,\mathrm{PolyLog}\left[4, e^{-i\,(t+x)}\right] - 2\,\mathrm{PolyLog}\left[4, -e^{i\,(t+x)}\right] + 2\,\mathrm{PolyLog}\left[4, e^{i\,(t+x)}\right]\right) + \mathrm{Cos}[t]\,\mathrm{Sin}[x]$

In[23]:= `Plot3D[v[x, t], {x, 0, Pi}, {t, 0, 2}, AxesLabel → {x, t, v}]`

Out[23]=